

Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres

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We construct higher dimensional quantum Hall systems based on fuzzy spheres. It is shown that fuzzy spheres are realized as spheres in colored monopole backgrounds. The space noncommutativity is related to higher spins which is originated from the internal structure of fuzzy spheres. In $2k$ -dimensional quantum Hall systems, Laughlin-like wave function supports fractionally charged excitations, $q = m^{-\frac{1}{2}k(k+1)}$ (m is odd). Topological objects are $(2k-2)$ -branes whose statistics are determined by the linking number related to the general Hopf map. Higher dimensional quantum Hall systems exhibit a dimensional hierarchy, where lower dimensional branes condense to make higher dimensional incompressible liquid.

Zhang and Hu have succeeded to construct a four dimensional generalization of quantum Hall (QH) systems [1]. Their systems have attracted much attention from physicists in various area [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Bernevig et al. also constructed eight dimensional “spinor” and “vector” QH systems [3]. They are based on the Hopf map related to the division algebra. Karabali and Nair have found another way to make higher dimensional QH systems based on $\mathbb{C}P$ manifolds [4]. Their construction originally had nothing to do with the Hopf map, but includes the four dimensional and the eight dimensional “vector” QH liquid. $\mathbb{C}P^3$ corresponds to the four dimensional QH liquid [4], while $\mathbb{C}P^7$ corresponds to the eight dimensional “vector” QH liquid [3].

One of the interesting features of QH systems is the appearance of the noncommutative geometry. Coordinates of electrons are noncommutative on the lowest Landau level. It is well known QH physics in two dimension is governed by noncommutative structure [16]. In Zhang and Hu’s work, a four dimensional sphere was not enough to construct a four dimensional QH system but an extra isospin space is needed, where the need of the isospin space was recognized heuristically. From the noncommutative geometrical point of view, the need of the extra isospin space is naturally understood. The noncommutative structure applies symplectic manifolds, which allow Poisson structure. $\mathbb{C}P$ manifolds are symplectic manifolds, while S^{2k} , $k \geq 2$ are not [17]. Therefore $\mathbb{C}P$ manifolds are enough to become higher dimensional QH systems, while S^{2k} , $k \geq 2$ need an extra isospin space to incorporate the noncommutative structure.

Therefore it seems reasonable to construct another kind of higher dimensional QH systems based on “symplectic spheres”, namely fuzzy spheres, instead of “ordinary spheres”. The general even dimensional QH systems have been analyzed in [15]. However the role of noncommutativity has not been clarified yet. In this article, we construct higher dimensional QH liquid based on fuzzy spheres. Our higher dimensional quantum liquid naturally incorporates the four dimensional and the eight dimensional “spinor” QH liquid.

Higher dimensional fuzzy spheres S_F^{2k} , $k \geq 2$ have some characteristic properties which cannot be seen in two dimensional fuzzy spheres. They are topological equivalent to the coset $SO(2k+1)/U(k)$ in a continuum limit, and have internal fiber spaces which form a lower dimensional fuzzy spheres [5, 18]. Therefore, the dimension of the fuzzy $2k$ -sphere is not $2k$ but $k(k+1)$. $SO(2k+1)/U(k)$ is a symplectic manifold, where the noncommutative structure can be incorporated.

The stabilizer group of the $SO(2k)$ spinor is $U(k)$. The coset $SO(2k)/U(k)$ is isomorphic to the configuration space of the $SO(2k)$ spinor. Therefore, it is promising to construct a projection,

$$S_F^{2k} \rightarrow S^{2k}, \quad (1)$$

with the use of Hopf spinor Ψ on S^{2k} . The explicit projection is established as,

$$\Psi \rightarrow \frac{X_a}{R} = \Psi^\dagger \Gamma_a \Psi, \quad \Psi^\dagger \Psi = 1, \quad (2)$$

where Γ_a ’s are Clifford algebra in $(2k+1)$ -dimensional space. X_a ’s are coordinates on S^{2k} , $\sum_{a=1}^{2k+1} X_a^2 = R^2$; the Hopf spinor Ψ is a coordinate on the manifold $U(1) \otimes S_F^{2k}$ as we shall see below.

The Clifford algebra Γ_a ’s are explicitly represented as,

$$\begin{aligned} \Gamma^i &= \begin{pmatrix} 0 & i\gamma^i \\ -i\gamma^i & 0 \end{pmatrix}, \quad i = 1, \dots, 2k-1 \\ \Gamma^{2k} &= \begin{pmatrix} 0 & 1_{2^{k-1} \times 2^{k-1}} \\ 1_{2^{k-1} \times 2^{k-1}} & 0 \end{pmatrix}, \\ \Gamma^{2k+1} &= \begin{pmatrix} 1_{2^{k-1} \times 2^{k-1}} & 0 \\ 0 & -1_{2^{k-1} \times 2^{k-1}} \end{pmatrix}. \end{aligned} \quad (3)$$

The Hopf spinor $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$, can be constructed in the following way,

$$\begin{aligned} \Psi_+ &= \sqrt{\frac{R + X_{2k+1}}{2R}} \Psi^{(2k-2)}, \\ \Psi_- &= \frac{1}{\sqrt{2R(R + X_{2k+1})}} (X_{2k} - iX_i\gamma^i) \Psi^{(2k-2)}, \end{aligned} \quad (4)$$

where γ^i 's are Clifford algebra in $(2k - 1)$ -dimensional space. Ψ is constructed by adding the $2k$ degrees of freedom X_a , $a = 1, 2, \dots, 2k + 1$, which is the coordinate on S^{2k} , to $\Psi^{(2k-2)}$. “Subcomponent” $\Psi^{(2k-2)}$ is also comprised of $(2k - 4)$ dimensional spinor $\Psi^{(2k-4)}$ just like (4). Similarly $\Psi^{(2k-4)}$ is comprised of $\Psi^{(2k-6)}$. Thus, we have such a sequence,

$$\Psi \rightarrow \Psi^{(2k-2)} \rightarrow \Psi^{(2k-4)} \rightarrow \dots \rightarrow \Psi^{(2)}. \quad (5)$$

The normalization $\Psi^\dagger \Psi = 1$ stems from a subcomponent normalization $\Psi^{(2k-2)\dagger} \Psi^{(2k-2)} = 1$, eventually from the minimal subcomponent normalization $\Psi^{(2)\dagger} \Psi^{(2)} = 1$. Since the Hopf spinor Ψ is constructed by adding the degrees of freedom of subcomponent spheres to $\Psi^{(2)}$, it has such degrees of freedom; $2k + (2k - 2) + \dots + 2 + 1 = k(k + 1) + 1$. It is equal to $S_F^{2k} \otimes U(1)$ degrees of freedom as it should be.

The holonomy group of the base manifold S^{2k} is $SO(2k)$. Then, the spinor which lives on S^{2k} is described as a section of the $Spin(2k)$ bundle on S^{2k} . Berry connection A , which is obtained from the Hopf spinor as $\Psi^\dagger d\Psi = \Psi^{(2k-2)\dagger} \cdot iA_a dX_a \cdot \Psi^{(2k-2)}$, is actually a $Spin(2k)$ connection,

$$A_\mu = -\frac{1}{R(R + X_{2k+1})} \Sigma_{\mu\nu}^+ X_\nu, \quad A_{2k+1} = 0, \quad (6)$$

where $\Sigma_{\mu\nu}$'s are $Spin(2k)$ generators,

$$\Sigma_{\mu\nu} = -i\frac{1}{4}[\Gamma_\mu, \Gamma_\nu] = \begin{pmatrix} \Sigma_{\mu\nu}^+ & 0 \\ 0 & \Sigma_{\mu\nu}^- \end{pmatrix}, \quad \mu, \nu = 1, \dots, 2k, \quad (7)$$

in detail, $\Sigma_{\mu\nu}^\pm = \{\Sigma_{ij}^\pm, \Sigma_{2k,i}^\pm\} = \{-i\frac{1}{4}[\gamma_i, \gamma_j], \mp\frac{1}{2}\gamma_i\}$. We locate an $Spin(2k)$ colored monopole at the center of S^{2k} to be compatible with the geometrical $SO(2k)$ holonomy, which generates the $Spin(2k)$ gauge connection on S^{2k} . We identify this monopole gauge connection with the spinor geometrical connection. Consequently, the magnitude of the spin I , in other words $SO(2k+1)$ spinor irreducible representation index, is related to the monopole charge g .

The colored monopole has been studied in [19, 20], which is transformed to a $Spin(2k)$ colored instanton on $2k$ -dimensional Euclidean space by a stereographic projection. Such an instanton configuration satisfies a higher dimensional dual equation and its stability is guaranteed by the homotopy mapping, $\pi_{2k-1}(SO(2k)) = Z$ [21]. The topological number is described by the k -th Chern number I , physically which corresponds to the colored monopole charge g as $g = \frac{1}{2}I$, $I \in \text{integer}$. For instance, the gauge connection (6) yields a unit element of the Chern number, or the monopole charge $1/2$. The monopole whose charge is $g = I/2$ is easily obtained from (6) by using the $(0, \dots, 0, I)$ representation of $Spin(2k+1)$. Due to the gauge and geometrical holonomy identification, the monopole charge is equivalent to the magnitude of the particle spin. Similarly, the

$Spin(2k)$ colored monopole generators can be regarded as the particle spins.

We construct higher dimensional QH systems based on this setup. The particle Hamiltonian is

$$H = \frac{\hbar^2}{2MR^2} \sum_{a < b} \Lambda_{ab}^2, \quad (8)$$

where Λ_{ab} is the covariant particle angular momentum, $\Lambda_{ab} = -i(X_a D_b - X_b D_a)$. D_a is the covariant momentum, $D_a = \partial_a + iA_a$. The algebraic relation of the covariant particle angular momentum Λ_{ab} is

$$[\Lambda_{ab}, \Lambda_{cd}] = i[\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}] - i[X_a X_c F_{bd} + X_b X_d F_{ac} - X_b X_c F_{ad} - X_a X_d F_{bc}]. \quad (9)$$

Λ_{ab} 's do not satisfy the $SO(2k+1)$ algebra, due to the existence of the monopole field strength F_{ab} .

Monopole angular momentum is $R^2 F_{ab}$, whose explicit form is $R^2 F_{\mu\nu} = -(X_\mu A_\nu - X_\nu A_\mu - \Sigma_{\mu\nu}^+)$, $R^2 F_{\mu 2k+1} = (R + X_{2k+1})A_\mu$. They satisfy same algebra as in (9) with the replacement $\Lambda_{ab} \rightarrow R^2 F_{ab}$. The magnitude of the monopole angular momentum is equal to the $Spin(2k)$ quadratic Casimir, $R^4 F_{ab}^2 = \Sigma_{\mu\nu}^{+2}$. The particle angular momentum Λ_{ab} is parallel to the tangent space of the base manifold S^{2k} , while the monopole angular momentum $R^2 F_{ab}$ is orthogonal to it. Therefore they are orthogonal in each other, $\Lambda_{ab} \cdot R^2 F_{ab} = R^2 F_{ab} \cdot \Lambda_{ab} = 0$.

The total angular momentum is the sum of the particle and the monopole angular momentum, $L_{ab} = \Lambda_{ab} + R^2 F_{ab}$, in detail, $L_{\mu\nu} = L_{\mu\nu}^{(0)} + \Sigma_{\mu\nu}^+$, $L_{\mu 2k+1} = L_{\mu 2k+1}^{(0)} + RA_\mu + X_\mu A_{2k+1}$, which satisfy the $SO(2k+1)$ algebra.

The particle Hamiltonian (8) commutes with the total angular momentum L_{ab} due to the $SO(2k+1)$ symmetry in the system. It is rewritten as $H = \frac{1}{2MR^2} \sum_{a < b} (L_{ab}^2 - R^4 F_{ab}^2)$, where the orthogonality of the particle and the monopole angular momentum was used. The eigenenergy of $Spin(2k+1)$ representation $(n, 0, \dots, 0, I)$ is

$$E(n, I) = \frac{\hbar^2}{2MR^2} [n^2 + n(I + 2k - 1) + \frac{1}{2}Ik], \quad (10)$$

where $n, I = 0, 1, 2, \dots$. n indicates Landau levels, while I determines the degeneracies in each Landau level.

The lowest Landau level (LLL) corresponds to $n = 0$. The energy and the degeneracy are given as

$$E_{LLL} = \frac{\hbar^2}{2M} \frac{I}{2R^2} k, \quad (11)$$

$$d(I) = \frac{(I + 2k - 1)!!}{(2k - 1)!!(I - 1)!!} \prod_{l=1}^{k-1} \frac{(I + 2l)!!}{(I + l)!(2l)!} \approx I^{\frac{1}{2}k(k+1)}. \quad (12)$$

The increase of the monopole charge $I/2$ induces the number of Dirac flux quanta or states as in (12). The eigenstates in the LLL are given as the $SO(2k+1)$ spinors $\{\Phi_\alpha\} \in (0, \dots, 0, I)$, which is constructed from the I -fold symmetric products of the Hopf spinor. It is because the Hopf spinor is the $SO(2k+1)$ fundamental

spinor $(0, \dots, 0, 1)$, which can be explicitly proven as $L_{ab}^2 \Psi_{\pm} = \Sigma_{ab}^{\pm 2} \Psi_{\pm}$ with the use of (4),(7).

Slater antisymmetric wavefunction for many particles is given as

$$\Phi_{Slater} = \sum_{\alpha_1, \dots, \alpha_N} \epsilon_{\alpha_1, \dots, \alpha_N} \Phi_{\alpha_1}(x_1) \dots \Phi_{\alpha_N}(x_N), \quad (13)$$

where N is the number of particles. The Laughlin-like wave function is obtained from (13) as

$$\Phi_{Laughlin} = \Phi_{Slater}^m. \quad (14)$$

The power m should be taken an odd integer to keep the antisymmetry of electrons.

The Slater function (13) has a property of incompressibility. It can be observed from the radial distribution function,

$$\begin{aligned} g(x, x') &= \int dx_3 \dots dx_N |\Phi_{Slater}(x, x', x_3, \dots, x_N)|^2 \\ &\approx 1 - \prod_{l=1}^k \exp\left(-\frac{1}{2\ell_B^2} (X_{(2l)}^\mu - X_{(2l)}^{'\mu})^2\right), \end{aligned} \quad (15)$$

where $x = (X_{(2)}, X_{(4)}, \dots, X_{(2k)})$; $X_{(2l)}$ represents the coordinates on the subcomponent sphere S^{2l} . In the second line, we have expanded the equation around the “north pole”; $X_{(2)}^3, X_{(4)}^5, \dots, X_{(2k)}^{2k+1} \approx R$. The radial distribution function converges exponentially to unity due to the strong suppression of density fluctuation, which is a typical property of the incompressible liquid [22]. Intriguingly, our liquid exhibits the incompressibility not only in the $2k$ -dimensional base space but also in each of the subcomponent lower dimensional spaces. This suggests that the higher dimensional QH liquid consists of lower dimensional incompressible liquid. We will revisit this later.

The thermodynamic limit naively corresponds to $I, R \rightarrow \infty$. To make the energy (10) finite, we take such a limit keeping the magnetic length $\ell_B = R\sqrt{\frac{2}{I}}$ finite. The filling factor ν is

$$\nu = \frac{N}{d(I)} \approx m^{-\frac{1}{2}k(k+1)}, \quad (16)$$

where $N = d(I)$. The density behaves as $\rho = N/V \approx d(I)/R^{k(k+1)} \approx \ell_B^{-k(k+1)}$, which is finite in the thermodynamic limit as it should be. Due to (16), Laughlin-like wave function supports excitations, whose charge is fractional, $q = m^{-\frac{1}{2}k(k+1)}$.

Around the “north pole”, the higher dimensional QH Lagrangean is reduced to

$$L = \frac{dX^\mu}{dt} A_\mu - V, \quad (17)$$

where we have added a confining potential term V . The gauge field is $A_\mu = -\frac{1}{\ell_B^2} \Sigma_{\mu\nu}^+ X_\nu$. $\Sigma_{\mu\nu}^+$'s can be expanded by a linear combination of the $Spin(2k)$ generators, $t^a, a = 1, 2, \dots, 2k^2 - 3k + 1$ as $\Sigma_{\mu\nu}^+ = (2k-1)\eta_{\mu\nu}^a t^a$.

$\eta_{\mu\nu}^a$ is an expansion coefficient, namely a generalized t'Hooft symbol. $(2k-1)$ is for convention. This expansion is always possible because it is just a recombination of basis. For example, at $k=2$, t^a 's are $SU(2)$ spin generators, at $k=3$, $SU(4)$ spin generators.

Canonical momentum reads $P_\mu = -\frac{1}{\ell_B^2} \Sigma_{\mu\nu}^+ X_\nu$. The coordinate and the momentum are orthogonal, due to the antisymmetry of the $SO(2k)$ generator, which physically corresponds to a higher dimensional cyclotron motion.

By imposing the canonical quantization condition, we obtain noncommutative algebra in higher dimensional QH systems,

$$[X_\mu, X_\nu] = -2i\ell_B^2 \eta_{\mu\nu}^a t^a, \quad (18)$$

which is consistent with the algebra obtained from the higher dimensional fuzzy spheres [18]. Since $\{t^a\}$ construct S_F^{2k-2} algebra, it is found that the noncommutativity of coordinates is related to the lower dimensional fuzzy sphere S_F^{2k-2} [5]. It is natural to regard this lower dimensional fuzzy sphere as an internal space. Therefore, electrons in the higher dimensional QH systems have effective higher spins. Thus, two important quantities, the space noncommutativity and the $Spin(2k)$ group spin, are related by the “fundamental length” ℓ_B as in (18). The equation of motion reads

$$\frac{d}{dt} X_\mu = -i[X_\mu, H] = -2\ell_B^2 \eta_{\mu\nu}^a t^a \frac{\partial V}{\partial X_\nu}. \quad (19)$$

The Hall current $\frac{d}{dt} X_\mu$ along a spin direction is determined by this equation, which is orthogonal to the “electric field” $-\frac{\partial V}{\partial X_\mu}$ as expected.

The subcomponent spheres S^{2k-2} on the base manifold S^{2k} are regarded as $(2k-2)$ -branes. $(2k-2)$ -branes move on the $2k$ -dimensional base space. Then, the total dimension of the space-time where $(2k-2)$ -brane exists is to be considered as $(4k-1)$. The statistics of $(2k-2)$ -branes is determined by the general Hopf map $\pi_{4k-1}(S^{2k}) = \mathbb{Z}, k \in \mathbb{N}$ [23]. (Strictly speaking, there sometimes appears an extra discrete group from the torsion part which we have omitted in the RHS. For instance, $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$. However it is not relevant in the following discussion.) The topological number represents the linking number of the $(2k-2)$ -branes in the $(4k-1)$ -dimensional space-time [24]. They are higher dimensional generalizations of the anyonic quasiparticles in two dimensional QH systems, whose statistics is determined by the braid group in 3-dimension [25].

Fractional QH states respect the Haldane-Halperin hierarchy [26]. In the dual picture, this hierarchy is understood as a formation of new Laughlin states by quasi-particles or 0-branes [27]. Here, we see that QH systems also exhibit another kind of hierarchy, namely a dimensional hierarchy. For instance, 0-branes which live on S^2 construct the two dimensional QH liquid. In general, on the subsphere S^{2l} , each $(2l-2)$ -brane occupies an

area of ℓ_B^{2l} . The number of the $(2l - 2)$ -brane on S^{2l} is $R^{2l}/\ell_B^{2l} \approx I^l$. These I^l $(2l - 2)$ -branes condense to make new $2l$ -dimensional incompressible liquid. Iteratively, the lower dimensional incompressible liquid condenses to make higher dimensional incompressible liquid. As a result, $2k$ -dimensional QH liquid is made of $I^{\frac{1}{2}(k-1)k}$ 2 -dimensional incompressible liquid. The higher dimensional filling factor (16) is naturally interpreted in this context. The filling factor on the subcomponent sphere S^{2l} is $I^l/(mI)^l = m^{-l}$. The filling factor ν in the total S_F^{2k} space is obtained by a product of the each filling factor in the subcomponent spheres,

$$\nu = \frac{1}{m} \cdot \frac{1}{m^2} \cdot \frac{1}{m^3} \cdots \frac{1}{m^k} = m^{-\frac{1}{2}k(k+1)}. \quad (20)$$

Thus, the higher dimensional filling factor represents the dimensional condensation of lower dimensional QH liquid [Fig.1]. This dimensional hierarchy implies an inti-

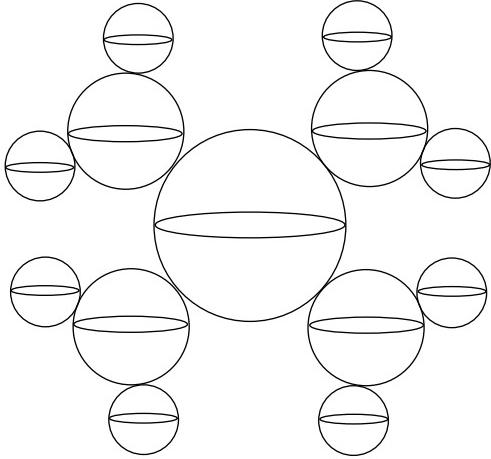


FIG. 1: Lower dimensional branes iteratively condense to make higher dimensional incompressible liquid.

mate connection between the QH systems and the matrix models. In the matrix theory picture, the higher dimensional D-brane is comprised of lower dimensional D-branes [28, 29]. The general filling factor is given by the combination of the dimensional and the Haldane-Halperin hierarchy,

$$\nu = \frac{1}{m \pm \frac{1}{2p_1 \pm \frac{1}{2p_2 \pm \cdots \frac{1}{2p_f}}}} \cdot \frac{1}{m^2 \pm \frac{1}{(2p_1)^2 \pm \frac{1}{(2p_2)^2 \pm \cdots \frac{1}{(2p_f)^2}}}} \cdots \frac{1}{m^k \pm \frac{1}{(2p_1)^k \pm \frac{1}{(2p_2)^k \pm \cdots \frac{1}{(2p_f)^k}}}}, \quad (21)$$

where in each space dimension, branes exhibit generalized a Haldane-Halperin hierarchy.

The edge of the $2k$ -dimensional QH liquid is S^{2k-1} , whose symmetry group is $SO(2k)$. As in the four dimensional case [2], there will also appear various $Spin(2k)$ higher spin particles as dipoles due to the nontrivial

$Spin(2k)$ composition rule. The edge action is described by the Wess-Zumino-Witten model [30]. Therefore, our edge states will arise as excitations in the $Spin(2k)$ Wess-Zumino-Witten model. We will report the detailed analysis of this model in the forth coming paper.

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